Periodic matrix: A square matrix A is called periodic matrix of order p if $A^{p+1} = A$

Example: $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is said to be of period 2

Solution: we have to show that A is of period 2 i.e. $A^{2+1} = A \Rightarrow A^3 = A$

So $A^2 = A.A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} . \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 1\times 1 + (-2)(-3) + (-6)\times 2 & 1\times(-2) + (-2)(2) + (-6)\times 0 & 1\times(-6) + (-2)\times 9 + (-6)\times(-3) \\ (-3)\times 1 + 2\times(-3) + 9\times 2 & (-3)\times(-2) + 2\times 2 + 9\times 0 & (-3)\times(-6) + 2\times 9 + 9\times(-3) \\ 2\times 1 + 0\times(-3) + (-3)\times 2 & 2\times(-2) + 0\times 2 + (-3)\times 0 & 2\times(-6) + 0\times 9 + (-3)\times(-3) \end{bmatrix}$$

$\Rightarrow A^2 = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}$

Then $A^3 = A^2.A = \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix} . \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$

$\Rightarrow A^3 = \begin{bmatrix} -5+18-12 & 10-12+0 & 30-54+18 \\ 9-30+18 & -18+20+0 & -54+90-27 \\ -4+12-6 & 8-9+0 & 24-36+9 \end{bmatrix} \Rightarrow A^3 = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \Rightarrow A^3 = A$

Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is periodic with period 4.

Solution

$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix};$

$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = I;$

$A^5 = A^4 \cdot A = IA = A$

Hence A is periodic and $\mathcal{P}(A) = 4$.

Exercises:

(i)	show that, $A = \begin{bmatrix} 2 & 5 & 14 \\ 1 & 3 & 8 \\ -1 & -2 & -6 \end{bmatrix}$ is of period 3	
(ii)	show that, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is of period 5	

Nilpotent matrix: A square matrix A is called nilpotent if $A^p = 0$, p is the degree of nilpotency.

Example 17. Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ $A^3 = O$ i.e., $A^k = O$

Here, $k = 3$

Hence, the matrix A is nilpotent of order 3.

Exercise:

Nilpotent of order 2	nilpotent of order 3	
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ $A = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix}$	$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}$	$A = \begin{bmatrix} 2 & 2 & 2 & -3 \\ 6 & 1 & 1 & -4 \\ 1 & 6 & 1 & -4 \\ 1 & 1 & 6 & -4 \end{bmatrix}$ is nilpotent of order 4.